

Anomalously light states in super-Yang–Mills Chern–Simons theory

J.R. Hiller,^a S.S. Pinsky,^b U. Trittmann^b

^a *Department of Physics
University of Minnesota Duluth
Duluth, MN 55812*

^b *Department of Physics
Ohio State University
Columbus, OH 43210, USA*

Abstract

Inspired by our previous finding that supersymmetric Yang–Mills–Chern–Simons (SYM-CS) theory dimensionally reduced to 1+1 dimensions possesses approximate Bogomol’nyi–Prasad–Sommerfield (BPS) states, we study the analogous phenomenon in the three-dimensional theory. Approximate BPS states in two dimensions have masses which are nearly independent of the Yang–Mills coupling and proportional to their average number of partons. These states are a reflection of the exactly massless BPS states of the underlying pure SYM theory. In three dimensions we find that this mechanism leads to anomalously light bound states. While the mass scale is still proportional to the average number of partons times the square of the CS coupling, the average number of partons in these bound states changes with the Yang–Mills coupling. Therefore, the masses of these states are not independent of the coupling. Our numerical calculations are done using supersymmetric discrete light-cone quantization (SDLCQ).

1 Introduction

It is extremely interesting to look for mechanisms in quantum field theory that give rise to anomalously light bound states, since these are the states we first see as we investigate new phenomena. Of course, spontaneous chiral symmetry breaking is the most profound example of these mechanisms. The underlying chiral symmetry of QCD gives rise to exactly massless bound states, the Goldstone bosons, which acquire masses from small mass terms for the constituent quarks which in turn break the symmetry.

What we are suggesting here is that there may be a similar mechanism at work in supersymmetric theories. $\mathcal{N} = 1$ pure supersymmetric Yang–Mills (SYM) theory in 1+1 and 2+1 dimensions has massless BPS states [1, 2]. Recently [3, 4] we created an effective mass for the constituents of a SYM theory in 1+1 dimensions by adding a dimensionally reduced Chern–Simons (CS) term [5] to the theory. Although such a term does not break the supersymmetry, the theory with a CS term does not have BPS states. We found, however, that this theory has approximate BPS states, i.e. states whose masses are nearly independent of the Yang–Mills (YM) coupling. Furthermore, at large YM coupling the masses of these states are equal to the square of the CS coupling times the average number of particles.

In this paper we will investigate the analog of these states in the (2+1)-dimensional theory. We will show that in 2+1 dimensions these states persist, although with some significant changes. We will see that in 2+1 dimensions the masses of these states are not independent of the YM coupling as they are in 1+1 dimensions. This is primarily because the average number of particles in these states grows with the coupling. However, the masses of these states do not grow as fast as the masses of the other bound states, and, therefore, at strong coupling these states are anomalously light.

The numerical method we will use is SDLCQ (Supersymmetric Discrete Light-Cone Quantization). This is a numerical method that can be used to solve any theory with enough supersymmetry to be finite. The central point of this method is that using DLCQ [6, 7] we can construct a finite dimensional representation of the superalgebra [8]. From this representation of the superalgebra, we construct a finite-dimensional Hamiltonian which we diagonalize numerically. We repeat the process for larger and larger representations and extrapolate the solutions to the continuum.

In constructing the discrete approximation we drop the longitudinal zero-momentum mode. For some discussion of dynamical and constrained zero modes, see the review [7] and previous work [9]. Inclusion of these modes would be ideal, but the techniques required to include them in a numerical calculation have proved to be difficult to develop, particularly because of nonlinearities. For DLCQ calculations that can be compared with exact solutions, the exclusion of zero modes does not affect the massive spectrum [7]. In scalar theories it has been known for some time that constrained zero modes can give rise to dynamical symmetry breaking [7], and work continues on the role of zero modes and near zero modes in these theories [10]. Dropping the zero modes in CS theory does result in the loss of many of the interesting aspects of CS theory, most notably the quantization of the CS coupling.

One particularly interesting property that *will* be preserved is the fact that the CS term simulates a mass for the theory. This property is interesting because reduced $\mathcal{N} = 1$ SYM theories are very stringy theories. The low-mass states are dominated by

Fock states with many constituents, and as the size of the superalgebraic representation is increased, states with lower masses and more constituents appear [1, 2, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. The connection between string theory and supersymmetric gauge theory leads one to expect this type of behavior. However, these gauge theories are not very QCD-like, and ultimately one might like to make a connection with the low-mass spectrum observed in nature. The effective mass supplied by the CS term makes the theory more like QCD.

The present Hamiltonian formalism provides us with the wave functions from which we construct the structure functions of the mass eigenstates. We will see that the structure functions of the approximate BPS states have unique shapes which might help to single them out experimentally.

The paper is structured as follows. In Sec. 2 we provide a summary of SYM-CS theory; much of this is given in [3] but is repeated here for completeness. Our numerical techniques are described in Sec. 3. An analysis of what should be expected for SYM-CS eigenstates is given in Sec. 4, and our results are presented and discussed in Sec. 5. Section 6 contains a summary of our conclusions and some discussion of interesting further investigations.

2 Supersymmetric Chern–Simons theory

The Lagrangian of (2+1)-dimensional $\mathcal{N} = 1$ SYM-CS theory is

$$\mathcal{L} = \text{Tr} \left(-\frac{1}{4} \mathcal{L}_{\text{YM}} + i \mathcal{L}_{\text{F}} + \frac{\kappa}{2} \mathcal{L}_{\text{CS}} \right), \quad (1)$$

where κ is the CS coupling and

$$\mathcal{L}_{\text{YM}} = F_{\mu\nu} F^{\mu\nu}, \quad (2)$$

$$\mathcal{L}_{\text{F}} = \bar{\Psi} \gamma_{\mu} D^{\mu} \Psi, \quad (3)$$

$$\mathcal{L}_{\text{CS}} = \epsilon^{\mu\nu\lambda} \left(A_{\mu} \partial_{\nu} A_{\lambda} + \frac{2i}{3} g A_{\mu} A_{\nu} A_{\lambda} \right) + 2 \bar{\Psi} \Psi. \quad (4)$$

The two components of the spinor $\Psi = 2^{-1/4} \begin{pmatrix} \psi \\ \chi \end{pmatrix}$ are in the adjoint representation of $U(N_c)$ or $SU(N_c)$. We will work in the large- N_c limit. The field strength and the covariant derivative are

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + ig[A_{\mu}, A_{\nu}], \quad D_{\mu} = \partial_{\mu} + ig[A_{\mu}, \quad]. \quad (5)$$

The supersymmetric variations of the fields leads to the supercurrent $Q^{(\mu)}$ in the usual manner via

$$\delta \mathcal{L} = \bar{\epsilon} \partial_{\mu} Q^{(\mu)}. \quad (6)$$

Light-cone coordinates in 2+1 dimensions are (x^+, x^-, x^{\perp}) where $x^+ = x_-$ is the light-cone time and $x^{\perp} = -x_{\perp}$. The totally anti-symmetric tensor is defined by $\epsilon^{+-2} = -1$. The variations of the three parts of the Lagrangian in Eq. (1) determine the (‘chiral’) components Q^{\pm} of the supercharge via Eq. (6) to be

$$\int d^2 x Q^{(+)} = \begin{pmatrix} Q^+ \\ Q^- \end{pmatrix} = -\frac{1}{2} \int d^2 x \epsilon^{\alpha\beta\lambda} \gamma_{\lambda} \gamma^+ \Psi F_{\alpha\beta}. \quad (7)$$

Explicitly they are

$$\begin{aligned} Q^- &= -i2^{3/4} \int d^2x \psi \left(\partial^+ A^- - \partial^- A^+ + ig[A^+, A^-] \right), \\ Q^+ &= -i2^{5/4} \int d^2x \psi \left(\partial^+ A^\perp - \partial^2 A^+ + ig[A^+, A^\perp] \right). \end{aligned} \quad (8)$$

One can convince oneself by calculating the energy-momentum tensor $T^{\mu\nu}$ that the supercharge fulfills the supersymmetry algebra

$$\{Q^\pm, Q^\pm\} = 2\sqrt{2}P^\pm, \quad \{Q^+, Q^-\} = -4P^\perp. \quad (9)$$

In order to express the supercharge in terms of the physical degrees of freedom, we have to use equations of motion, some of which are constraint equations. The equations of motion for the gauge fields are

$$-D_\nu F^{\nu\alpha} = \frac{\kappa}{2} \epsilon^{\alpha\nu\lambda} F_{\nu\lambda} + 2g\bar{\Psi}\gamma^\alpha\Psi. \quad (10)$$

For $\alpha = +$ this is a constraint for A^- . In light-cone gauge, $A^+ = 0$, the constraint reduces to

$$D_- A^- = \frac{1}{D_-} [(\kappa - D_2) D_- A^\perp + 2g\bar{\Psi}\gamma^+\Psi]. \quad (11)$$

The equation of motion for the fermion is $\gamma^\mu D_\mu \Psi = -i\kappa\Psi$. Expressing everything in terms of ψ and χ leads to the equations of motion

$$\sqrt{2}D_+\psi = (D_2 + \kappa)\chi, \quad (12)$$

$$\sqrt{2}D_-\chi = (D_2 - \kappa)\psi, \quad (13)$$

the second of which is a constraint equation. The constraint equations, (11) and (13), are used to eliminate A^- and χ .

We compactify in the x^\perp direction with compactification length L , meaning that transverse momentum modes are summed over a discrete set of values $2\pi n^\perp/L$. In order to have a finite matrix representation for the eigenvalue problem, we must truncate these sums at some fixed integers $\pm T$. The value of T defines a physical transverse cutoff $\Lambda_\perp = 2\pi T/L$; however, given this definition, T can be viewed as a measure of transverse resolution at fixed Λ_\perp .

At large N_c the expansions of the field operators $\phi \equiv A_\perp$ and ψ , in terms of creation and annihilation operators for the Fock basis, are

$$\begin{aligned} \phi_{ij}(0, x^-, x_\perp) &= \frac{1}{\sqrt{2\pi L}} \sum_{n^\perp=-\infty}^{\infty} \int_0^\infty \frac{dk^+}{\sqrt{2k^+}} \left[a_{ij}(k^+, n^\perp) e^{-ik^+x^- + i\frac{2\pi n^\perp}{L}x_\perp} + a_{ji}^\dagger(k^+, n^\perp) e^{ik^+x^- - i\frac{2\pi n^\perp}{L}x_\perp} \right], \\ \psi_{ij}(0, x^-, x_\perp) &= \frac{1}{2\sqrt{\pi L}} \sum_{n^\perp=-\infty}^{\infty} \int_0^\infty dk^+ \left[b_{ij}(k^+, n^\perp) e^{-ik^+x^- + i\frac{2\pi n^\perp}{L}x_\perp} + b_{ji}^\dagger(k^+, n^\perp) e^{ik^+x^- - i\frac{2\pi n^\perp}{L}x_\perp} \right]. \end{aligned}$$

From the field (anti-)commutators one finds

$$[a_{ij}(p^+, n_\perp), a_{lk}^\dagger(q^+, m_\perp)] = \{b_{ij}(p^+, n_\perp), b_{lk}^\dagger(q^+, m_\perp)\} = \delta(p^+ - q^+) \delta_{n_\perp, m_\perp} \delta_{il} \delta_{jk}. \quad (14)$$

The truncated supercharge Q^- can be written as

$$Q^- = gQ_{\text{SYM}}^-(T) + Q_{\perp}(T) + i\kappa Q_{\text{CS}}^-(T), \quad (15)$$

where

$$Q_{\perp}^-(T) = \frac{2^{3/4}\pi i}{L} \sum_{|n^{\perp}| \leq T} \int_0^{\infty} dk \frac{n^{\perp}}{\sqrt{k}} \left[a_{ij}^{\dagger}(k, n^{\perp}) b_{ij}(k, n^{\perp}) - b_{ij}^{\dagger}(k, n^{\perp}) a_{ij}(k, n^{\perp}) \right], \quad (16)$$

$$Q_{\text{CS}}^-(T) = 2^{-1/4}i \sum_{|n^{\perp}| \leq T} \int_0^{\infty} dk \frac{1}{\sqrt{k}} \left[a_{ij}^{\dagger}(k, n^{\perp}) b_{ij}(k, n^{\perp}) + b_{ij}^{\dagger}(k, n^{\perp}) a_{ij}(k, n^{\perp}) \right], \quad (17)$$

and

$$\begin{aligned} Q_{\text{SYM}}^-(T) = & \frac{i2^{-5/4}}{\sqrt{L\pi}} \sum_{|n_i^{\perp}| \leq T} \int_0^{\infty} dk_1 dk_2 dk_3 \delta(k_1 + k_2 - k_3) \delta_{n_1^{\perp} + n_2^{\perp}, n_3^{\perp}} \left\{ 2 \left(\frac{1}{k_1} + \frac{1}{k_2} - \frac{1}{k_3} \right) \right. \\ & \times \left[b_{ik}^{\dagger}(k_1, n_1^{\perp}) b_{kj}^{\dagger}(k_2, n_2^{\perp}) b_{ij}(k_3, n_3^{\perp}) + b_{ij}^{\dagger}(k_3, n_3^{\perp}) b_{ik}(k_1, n_1^{\perp}) b_{kj}(k_2, n_2^{\perp}) \right] \\ & + \frac{k_2 - k_1}{k_3 \sqrt{k_1 k_2}} \left[a_{ik}^{\dagger}(k_1, n_1^{\perp}) a_{kj}^{\dagger}(k_2, n_2^{\perp}) b_{ij}(k_3, n_3^{\perp}) - b_{ij}^{\dagger}(k_3, n_3^{\perp}) a_{ik}(k_1, n_1^{\perp}) a_{kj}(k_2, n_2^{\perp}) \right] \\ & + \frac{k_1 + k_3}{k_2 \sqrt{k_1 k_3}} \left[a_{ik}^{\dagger}(k_3, n_3^{\perp}) a_{kj}(k_1, n_1^{\perp}) b_{ij}(k_2, n_2^{\perp}) - a_{ik}^{\dagger}(k_1, n_1^{\perp}) b_{kj}^{\dagger}(k_2, n_2^{\perp}) a_{ij}(k_3, n_3^{\perp}) \right] \\ & \left. + \frac{k_2 + k_3}{k_1 \sqrt{k_2 k_3}} \left[b_{ik}^{\dagger}(k_1, n_1^{\perp}) a_{kj}^{\dagger}(k_2, n_2^{\perp}) a_{ij}(k_3, n_3^{\perp}) - a_{ij}^{\dagger}(k_3, n_3^{\perp}) b_{ik}(k_1, n_1^{\perp}) a_{kj}(k_2, n_2^{\perp}) \right] \right\}. \end{aligned} \quad (18)$$

The symmetric truncation with respect to positive and negative modes aids in retaining a reflection parity symmetry in the states. The light-cone energy is $(k_{\perp}^2 + m^2)/k^+$, and k_{\perp} behaves like a mass. Here we see that κ appears in a very similar way to k_{\perp} and therefore behaves in many ways like a mass.

When comparing the \perp and CS contributions to the supercharge, we see that we have a relative i between them. Thus the usual eigenvalue problem

$$2P^+P^-|\varphi\rangle = \sqrt{2}P^+(Q^-)^2|\varphi\rangle = \sqrt{2}P^+(gQ_{\text{SYM}}^- + Q_{\perp}^- + i\kappa Q_{\text{CS}}^-)^2|\varphi\rangle = M_n^2|\varphi\rangle \quad (19)$$

has to be solved by using fully complex methods.

We retain¹ the S -symmetry, which is associated with the orientation of the large- N_c string of partons in a state [21]. In a (1+1)-dimensional model this orientation parity is usually referred as a Z_2 symmetry, and we will follow that here. It gives a sign when the color indices are permuted

$$Z_2 : a_{ij}(k, n^{\perp}) \rightarrow -a_{ji}(k, n^{\perp}), \quad b_{ij}(k, n^{\perp}) \rightarrow -b_{ji}(k, n^{\perp}). \quad (20)$$

We will use this symmetry to reduce the Hamiltonian matrix size and hence the numerical effort. All of our states will be labeled by the Z_2 sector in which they appear. We will not attempt to label the states by their normal parity; in light-cone coordinates this is only an approximate symmetry. Such a labeling could be done in an approximate way, as was shown by Hornbostel [22], and might be useful for comparison purposes if at some point there are results from lattice simulations of the present theory.

¹We note that the CS term breaks transverse parity.

3 Numerical Methods

We convert the mass eigenvalue problem $2P^+P^-|M\rangle = M^2|M\rangle$ to a matrix eigenvalue problem by introducing a discrete P^- in a basis where P^+ and $P^\perp = 0$ are diagonal. As discussed in the Introduction, this is done in SDLCQ by first discretizing the supercharge Q^- and then constructing P^- from the square of the supercharge: $P^- = (Q^-)^2/\sqrt{2}$. We have already introduced a finite discretization in the transverse direction, characterized by the compactification scale L and cutoff or resolution T . To complete the discretization of the supercharge, we introduce discrete longitudinal momenta k^+ as fractions nP^+/K of the total longitudinal momentum P^+ . Here $n < K$ and K are positive integers. The number of partons is also bounded by K . The integer K determines the resolution of the longitudinal discretization and is known in DLCQ as the harmonic resolution [6]. The remaining integrals in Q^- are approximated by a trapezoidal form. The continuum limit in the longitudinal direction is then recovered by taking the limit $K \rightarrow \infty$. In constructing the discrete approximation we drop the longitudinal zero-momentum mode.

Our earliest SDLCQ calculations [16] were done using a code written for MATHEMATICA and performed on a PC. This code was rewritten in C++ for the work presented in [14] and has been substantially revised to reach higher resolutions [1]. The MATHEMATICA code continues to be useful for checking any changes in the C++ code.

To obtain the spectrum of the CS theory we solve the complex eigenvalue problem, Eq. (19). For the numerical evaluation we can exploit the structure of the supercharge

$$Q^- = \begin{pmatrix} 0 & A + iB \\ A^T - iB^T & 0 \end{pmatrix}, \quad (21)$$

where A and B are real matrices. The Hamiltonian has thus a simple decomposition into real and imaginary parts in the bosonic sector

$$P_{\text{boson}}^- = AA^T + BB^T + i(BA^T - AB^T), \quad (22)$$

and the fermionic sector

$$P_{\text{fermion}}^- = A^T A + B^T B + i(A^T B - B^T A). \quad (23)$$

We extract several of the lowest eigenstates by applying the Lanczos algorithm [23], as discussed in [1].

4 Setting the stage

To analyze the full SYM-CS theory, it is convenient to first study three subsets of the (2+1)-dimensional theory: the (1+1)-dimensional pure SYM theory ($T = 0$, $\kappa = 0$), the pure SYM theory in three dimensions ($T > 0$, $\kappa = 0$), and the (1+1)-dimensional SYM-CS theory ($T = 0$, $\kappa \neq 0$). We have previously studied each of these theories separately [19, 20, 1, 3].

Let us start with the dimensionally reduced pure SYM theory [19, 20], for which

$$Q^- = gQ_{\text{SYM}}^-(T = 0). \quad (24)$$

There are two unique properties that characterize this theory. As we increase the resolution, we find that there are new lower-mass states that appear, and this sequence of states appears to accumulate at $M^2 = 0$. In addition, there are massless BPS states. The dominant component of the wave function of the BPS states can be arranged to have $2, 3, \dots, K$ particles, i.e. up to the maximum number of particles allowed at resolution K . Therefore at resolution K there are $K - 1$ bosonic BPS states and $K - 1$ fermionic BPS states.

Next we focus on the case $T > 0$, $\kappa = 0$, which is the pure SYM theory in $2 + 1$ dimensions [1]. The supercharge Q^- becomes

$$Q^- = gQ_{\text{SYM}}^-(T) + Q_{\perp}^-(T). \quad (25)$$

Again there are a number of unique properties that characterize the theory. At small g and low energy we recover the (1+1)-dimensional theory. At higher energy we find a series of Kaluza–Klein states, which we discuss in detail elsewhere [24]. At large coupling one might expect that the transverse Q_{\perp}^-/g term would freeze out, and one would see states that are a reflection of the (1+1)-dimensional theory. While we see these states, they are only a small part of the spectrum. Q_{SYM}^- by itself wants to induce a large number of particles, as we discussed above. The contributions from gQ_{SYM}^- and Q_{\perp}^- are therefore minimized by a larger number of particles, each with a small transverse momentum. We therefore find that the average number of particles in the bound states grows with the coupling. The massless BPS states of the (1+1)-dimensional SYM theory persist in 2+1 dimensions, but now the number of particles in all of these states increases rapidly with the coupling. In Fig. 3 of Ref. [16] we displayed the average number of particles in the massless states as a function of the scaled coupling. At zero coupling we have the (1+1)-dimensional theory, and the states arrange themselves to have 2 through K particles.

If we set $T = 0$ we get SYM-CS theory reduced to two dimensions [3]. Here we find that the most important role of the CS term is to provide a mass for the constituents. This freezes out the long, lower-mass states that characterize (1+1)-dimensional SYM theory. Interestingly, however, the massless BPS states become massive approximate BPS states and have masses that are nearly independent of the YM coupling [4], as seen in Fig. 1(a) where we show the two lowest Z_2 -even approximate BPS states have squared masses of $4\kappa^2$ and $16\kappa^2$ at infinite YM coupling.

5 Numerical Results

5.1 DLCQ analysis

We start the discussion of our new results with a standard DLCQ analysis of the bound states at² $g = 0.5$. At each value of the resolution K and in each Z_2 sector we look at the ten lowest-energy bound states at each transverse resolution T . In order to extrapolate to infinite transverse momentum cutoff, we fit the curves of mass squared as a function of the inverse transverse cutoff with a linear function in $1/T$, for $T \geq 3$, as shown in Fig. 2(a). The intercepts of these curves are the transverse continuum results, which we plot as a function of $1/K$ in Fig. 2(b). Again we make a fit to the lowest

²Values of g are given in units of $\sqrt{4\pi^3/N_c L}$.

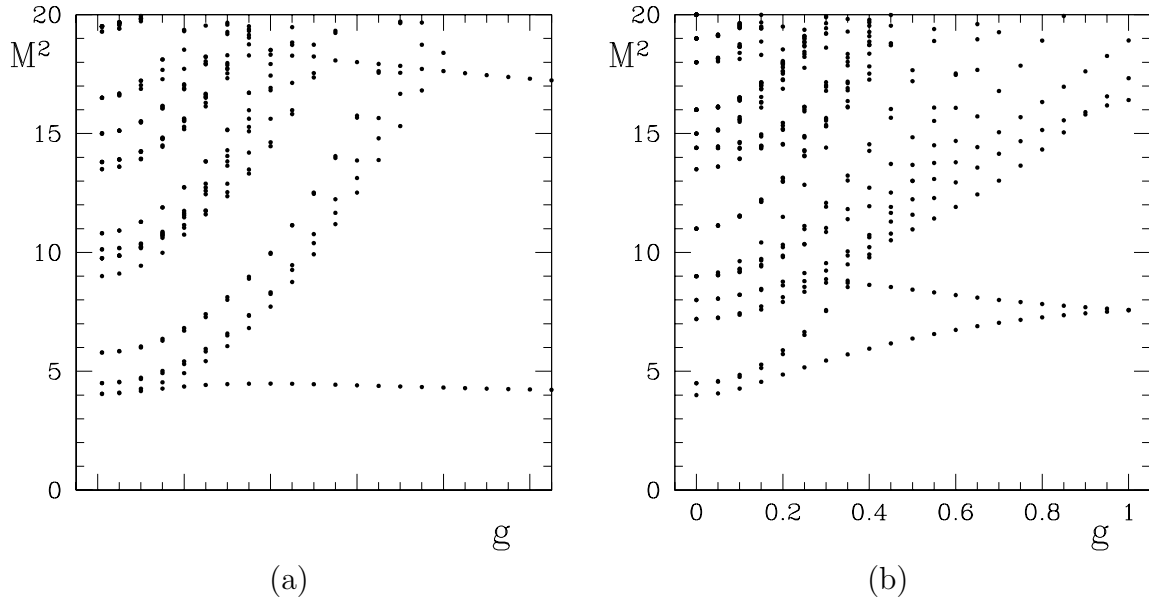


Figure 1: The Z_2 -even spectrum of SYM-CS theory as a function of the Yang-Mills coupling g in (a) 1+1 dimensions and (b) 2+1 dimensions. For (a), the longitudinal resolution is $K = 9$, M^2 is measured in units of κ^2 , and g is in units of $\kappa\sqrt{\pi/N_c}$. For (b), the longitudinal resolution is $K = 6$, the transverse resolution is $T = 1$, the Chern-Simons coupling is $\kappa = 2\pi/L$, M^2 is in units of $4\pi^2/L^2$, and g is in units of $\sqrt{4\pi^3/N_c L}$.

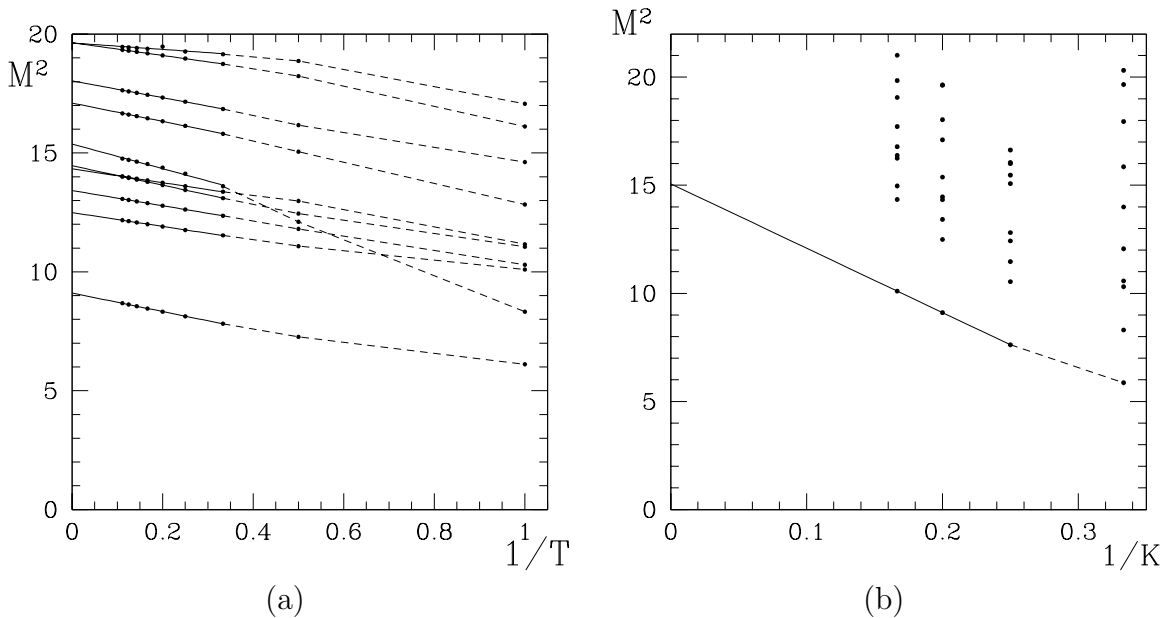


Figure 2: The Z_2 -even spectrum of (2+1)-dimensional SYM-CS theory as a function of (a) transverse resolution T and (b) longitudinal resolution K . The values in (b) have been extrapolated to $T = \infty$. The coupling strengths are $g = 0.5$, in units of $\sqrt{4\pi^3/N_c L}$, and $\kappa = 2\pi/L$.

mass state for $K \geq 4$, this time linear in $1/K$, and the intercept of this curve is the true continuum mass. We have not attempted to fit the $1/K$ behavior of the higher states, because we are mainly interested in the lowest state, which is the analog of the lowest approximate BPS state in two dimensions.

For the case $g = 0.1$ the average number of particles is well behaved, in the sense that it stays small. This allows for an additional approximation. Namely, we can restrict the number of partons per Fock state. This is a good approximation to the lowest ten states. Their average parton number is either two or three for even or odd Z_2 , respectively. This is clear since the states at small coupling are very close to the free states, which in turn are the discrete manifestations of the two and three particle continua starting at $M^2 = 4\kappa^2$ and $M^2 = 9\kappa^2$, where κ^2 is the effective parton mass. We restrict the number of partons in a state to four. This additional approximation allows us to carry the calculation to nine units of transverse resolution, $T = 9$, and nine units of longitudinal resolution, $K = 9$. The details of this analysis will be presented elsewhere [24]. At $g = 0.5$ we are unable to make this approximation and are only able to go to maximum resolutions of $K = 6$ and $T = 4$. The reason that this approximation breaks down is that, as we increase T and K , the average number of partons in a state increases substantially. When this number approaches K , we miss a significant part of the wave function in the calculation.

In the $Z_2 = -1$ bosonic sector the lowest state is a three-particle state at weak coupling. All the analysis of this sector goes through similarly to the $Z_2 = +1$ sector and will also be presented elsewhere [24].

5.2 Dependence of the spectrum on the Yang-Mills coupling

In Fig. 1(b) we show the coupling dependence of SYM-CS theory in 2+1 dimensions in the $Z_2 = +1$ sector. We will restrict ourselves to this sector in the discussion that follows. For numerical convenience the results presented here are at specific and low values of the longitudinal and transverse cutoffs K and T but at many values of the coupling. The point we want to make is that the phenomenon of approximate BPS states persists from two dimensions: some states appear to be nearly independent of the coupling at fixed T and K .

We can distinguish two regions in the low-lying spectrum. In the region of very small g we have a spectrum of almost free particles, and in the regime of larger g an approximate BPS state is clearly visible while the masses squared of the rest of the states grow like g^2 . In the intermediate region we see a large number of level crossings as the states rearrange themselves. This transition occurs around $g = 0.3$. We will refer to couplings above this cross-over point as being strong, because the spectrum has completely rearranged itself here.

We have investigated the spectrum at $g = 0.1$ and $g = 0.5$ as a function of K and T as typical representatives of weak and strong coupling, respectively. It is obvious that some of the states seen in Fig. 1(b) move differently as a function of K and T . At weak coupling we have the manifestation of a multi-particle continuum within a discrete approach, and the masses of the lowest states are very close together. As the coupling grows, a large gap develops between the lowest-mass state, which is an approximate BPS state, and the other states in the spectrum, as seen in Fig. 1(b). The anomalous behavior of the next to lowest state, which is the only state to decrease in mass, is a small transverse cutoff effect. When T is increased the gap is restored, as can be seen in Fig. 2(a): the mass of the next to lowest state at $T = 1$ increases rapidly as T grows, leaving a mass gap in the transverse continuum limit. The computing time required to produce a graph of the spectrum as a function of the coupling at large T is prohibitive, but we display the lowest masses in Table 1 at $g = 0.1$ and $g = 0.5$. We see that at $g = 0.1$ the lowest 10 states are more or less uniformly spaced from $M^2 = 4$ to $M^2 = 10$ in units of $4\pi^2/L^2$. This is what we would expect since at small coupling these states are closely related to the two-particle continuum states. At strong coupling, $g = 0.5$, we see that a large gap has developed between the state at $M^2 = 8.93$ and $M^2 = 13.14$. These results are obtained at resolution $K = 6$, but from Fig. 2(b) it appears that this gap is growing with the longitudinal resolution, and, therefore, in the continuum we expect it will be even larger.

We have studied the average number of particles of all the states in all of the sectors and a detailed discussion will be presented elsewhere [24]. Here it is interesting to note that for the state at $M^2 = 8.93$ we find an average parton number squared $\langle n^2 \rangle = 8.23$. This is close to the expected value based on the duality discussed in SYM-CS in 1+1 dimensions [4], where we found that the mass at infinite coupling is the average parton number squared at CS coupling unity. By contrast, the average parton number squared $\langle n^2 \rangle = 9.9$ of the second lightest state is far away from its mass squared $M^2 = 13.14$.

g	M^2									
0.1	4.36	4.85	4.93	7.41	7.47	8.35	8.36	9.34	9.43	9.48
0.5	8.93	13.14	13.88	14.92	15.10	15.62	15.82	17.38	18.35	19.76

Table 1: Mass squared in units of $4\pi^2/L^2$ of the lowest-mass states at $g = 0.1$ and $g = 0.5$, in units of $\sqrt{4\pi^3/N_c L}$, showing the development of a gap above the lowest state. Results in both cases are for resolutions $K = 6$ and $T = 4$ and for Chern–Simons coupling $\kappa = 2\pi/L$.

5.3 Structure Functions

In this section we will present a discussion of the structure functions of the approximate BPS states. A complete discussion of the structure functions of all the states will be presented elsewhere [24]. The structure functions are functions of both the longitudinal and transverse momentum and are defined to be the probability that a particle of type A will have a longitudinal momentum fraction $x = k^+/P^+$ and transverse momentum k^\perp . It is given as follows in terms of the light-cone wave function ψ :

$$\begin{aligned} \hat{g}_A(x, k^\perp) = & \sum_q \int_0^1 dx_1 \cdots dx_q \int_{-\infty}^{\infty} dk_1^\perp \cdots dk_q^\perp \delta\left(\sum_{i=1}^q x_i - 1\right) \delta\left(\sum_{j=1}^q k_j^\perp\right) \\ & \times \sum_{l=1}^q \delta(x_l - x) \delta(k_l^\perp - k^\perp) \delta_{A_l}^A |\psi(x_1, k_1^\perp; \dots x_q, k_q^\perp)|^2. \end{aligned} \quad (26)$$

We present the discrete version of the structure function as a function of the number of units of longitudinal momentum and the number of units of transverse momentum. No attempt has been made to extrapolate these curves.

In Fig. 3 we see the structure function of the approximate BPS state of the SYM-CS theory for $g = 0.1$ and $g = 0.5$. At weak coupling it has one localized peak in both variables. In fact this is true of all of the low-mass states of this theory at weak coupling, and the approximate BPS state does not appear in any way special. As we move to stronger coupling, however, all structure functions of the normal bound states remain peaked in n_\parallel , while the peak moves down to lower values of n_\parallel as the number of particles increases. In contrast we see that the BPS state is nearly flat in n_\parallel , although it has a small peak at $n_\parallel = 1$. This behavior is unique to this state.

At this point we do not have an understanding of why the approximate BPS state has this behavior. It is, however, clearly important that it does, if for no other reason than to give one a means to identify such a state experimentally.

6 Outlook

In our earlier work on dimensionally reduced SYM-CS theory [3, 4], we found that there were states whose properties are largely dictated by the properties of the massless BPS states of the underlying SYM theory. We saw that the masses of these states were nearly constant as functions of the YM coupling, and we therefore called these states approximate BPS states. The mass scale of these states is set by the CS coupling and

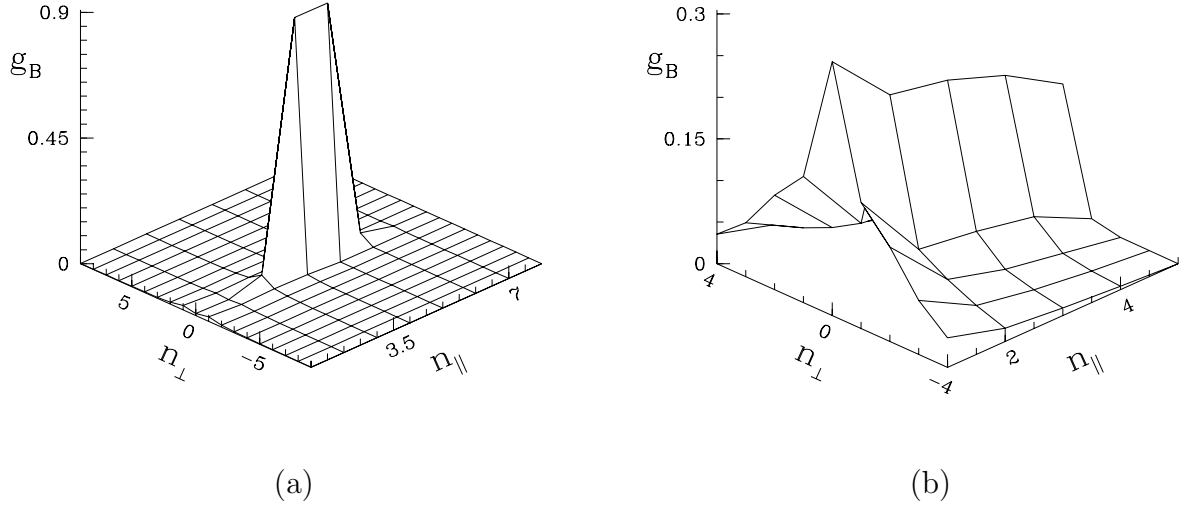


Figure 3: Structure function of the lowest approximate BPS state at (a) $g = 0.1$ and (b) $g = 0.5$, in units of $\sqrt{4\pi^3/N_c L}$. The numerical resolutions are $K = 6$ and $T = 4$. The value of the Chern–Simons coupling is $\kappa = 2\pi/L$.

the average number of partons. The binding energy is approximately zero, because of the underlying properties of the BPS state.

In this paper we have looked for the effect of these approximate BPS states in the full (2+1)-dimensional SYM-CS theory. We have looked at the low mass spectrum and found that at strong coupling there are states with an anomalously low mass. The mass scale for these states is set by the CS coupling and the average number of partons, similar to the (1+1)-dimensional case. These states are a reflection of the exact BPS states that exist in (2+1)-dimensional SYM theory. In addition, we examined the structure functions for these states. We find that they are different from the structure functions of the other states; they are nearly flat in longitudinal momentum.

It appears that we have found a new mechanism for generating low-mass bound states. There are two issues, however, that need to be addressed before speculating on the possibility that this mechanism might be realized in nature. First, we have not included any matter fields in this theory. There has been some work done with matter fields in SDLCQ [11], and one of our next projects will be to include them. Second, we have not included supersymmetry breaking. Instead, we added masses for the adjoint fields by including a CS term. The calculational technique we use, SDLCQ, relies heavily on the fact that we have an exactly supersymmetric theory, and it is essential, therefore, that we introduce masses without breaking the supersymmetry. Inclusion of supersymmetry breaking remains an outstanding problem in this numerical approach.

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References

- [1] J.R. Hiller, S. Pinsky, U. Trittman, Phys. Rev. D **64** (2001) 105027, hep-th/0106193.
- [2] J.R. Hiller, S. Pinsky, U. Trittman, Phys. Rev. D **63** (2001) 105017, hep-th/0101120.
- [3] J.R. Hiller, S.S. Pinsky, U. Trittman, Phys. Rev. D **65** (2002) 085046, hep-th/0112151.
- [4] J.R. Hiller, S.S. Pinsky, U. Trittman, submitted for publication, hep-th/0203162.
- [5] G.V. Dunne, “Aspects of Chern–Simons Theory,” Lectures at the 1998 Les Houches NATO Advanced Studies Institute, Session LXIX, *Topological Aspects of Low Dimensional Systems*, edited by A. Comtet *et al.*, pp. 177-263, (Springer–Verlag, Berlin, 2000), hep-th/9902115.
- [6] H.-C. Pauli, S.J. Brodsky, Phys. Rev. D **32** (1985) 1993; **32** (1985) 2001.
- [7] S.J. Brodsky, H.-C. Pauli, S.S. Pinsky, Phys. Rep. **301** (1998) 299, hep-ph/9705477.
- [8] Y. Matsumura, N. Sakai, T. Sakai, Phys. Rev. D **52** (1995) 2446.
- [9] F. Antonuccio, O. Lunin, S. Pinsky, S. Tsujimaru, Phys. Rev. D **60** (1999) 115006, hep-th/9811254.
- [10] J.S. Rozowsky, C.B. Thorn, Phys. Rev. Lett. **85** (2000) 1614, hep-th/0003301.
- [11] O. Lunin, S. Pinsky, Phys. Rev. D **63** (2001) 045019, hep-th/0005282.
- [12] S. Pinsky, U. Trittman, Phys. Rev. D **62** (2000) 087701, hep-th/0005055.
- [13] J.R. Hiller, O. Lunin, S. Pinsky, U. Trittman, Phys. Lett. B **482** (2000) 409, hep-th/0003249.
- [14] P. Haney, J.R. Hiller, O. Lunin, S. Pinsky, U. Trittman, Phys. Rev. D **62** (2000) 075002, hep-th/9911243.
- [15] O. Lunin, S. Pinsky, AIP Conf. Proc. **494** (1999) 140, hep-th/9910222.
- [16] F. Antonuccio, O. Lunin, S. Pinsky, Phys. Rev. D **59** (1999) 085001, hep-th/9811083.

- [17] F. Antonuccio, H.-C. Pauli, S. Pinsky, S. Tsujimaru, Phys. Rev. D **58** (1998) 125006, hep-th/9808120.
- [18] F. Antonuccio, O. Lunin, S. Pinsky, H.-C. Pauli, S. Tsujimaru, Phys. Rev. D **58** (1998) 105024, hep-th/9806133.
- [19] F. Antonuccio, O. Lunin, S. Pinsky, Phys. Rev. D **58** (1998) 085009, hep-th/9803170.
- [20] F. Antonuccio, O. Lunin, S.S. Pinsky, Phys. Lett. B **429** (1998) 327, hep-th/9803027.
- [21] D. Kutasov, Nucl. Phys. B **414** (1994) 33.
- [22] K. Hornbostel, Ph.D. thesis, SLAC Report No. 333, 1988.
- [23] C. Lanczos, J. Res. Nat. Bur. Stand. **45**, 255 (1950); J. Cullum and R.A. Willoughby, *Lanczos Algorithms for Large Symmetric Eigenvalue Computations* (Birkhauser, Boston, 1985), Vol. I and II.
- [24] J.R. Hiller, S. Pinsky, U. Trittman, manuscript in preparation.